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ANKENY, ARTIN AND CHOWLA CONJECTURE FOR EVEN GENERATORS

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In the paper [2] Ankeny, Artin and Chowla conjectured that if p is a prime such that $p \equiv 1 \pmod 4$ and $\varepsilon = \frac{t+u\sqrt{p}}{2}$ is the fundamental unit of the real quadratic field $K = Q\left(\sqrt{p}\right)$ then $p \nmid u$.

In this paper we prove that this conjecture is true for even generators t, u of the fundamental unit ε.

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Introduction. Let $K = Q\left(\sqrt{p}\right)$ be the quadratic number field with the prime number $p \equiv 1 \pmod 4$. Then it is well-known that the fundamental unit ε has the following form:

$$(1.1) \qquad \varepsilon = \frac{t+u\sqrt{p}}{2} > 1,$$

where the generators t, u are the same parity. Moreover, it is known that the norm $N(\varepsilon)$ of this unity is equal to -1,so $N(\varepsilon) = -1$. From this fact immediately follows that

$$(1.2) \qquad t^2 - pu^2 = -4,$$

From (1.2) follows that for the solution of the Ankeny, Artin and Chowla conjecture it suffices to consider the Diophantine equation (1.2).

We remember that Mordell [7] proved that if prime $p \equiv 5 \pmod 8$ then $u \equiv 0 \pmod p$ if and only ifg the Bernoulli number $B_{\frac{p-1}{4}} \equiv 0 \pmod p$.

This criterion for remaining primes $p \equiv 1 \pmod 4$ has been proved by Ankeny and Chowla in the paper [3].

Moreover, in the paper [4] Chowla remarked some intersting congruence relation connected with the class number h of the field $K = Q\left(\sqrt{p}\right), p \equiv 1 \pmod 4$.Namely, we have

(1.3) $\left(\frac{p-1}{2}\right)! \equiv (-1)^{\frac{h+1}{2}} \cdot \frac{t}{2} \pmod p$.

Another criteria cinnected with AAC conjecture has been given by Agoh [1] and by Yokoi [10]. In the paper [8],[9] Sheighorn obtained interesting connections between fundamental solution $\langle x_0, y_0\rangle$ of the negative Pell's equation

(1.4) $x^2 - py^2 = -1$

with $p \equiv 1 \pmod 4$ and the manner of reflection lines on the modular surface and the $\sqrt{p}$ Riemann surface.

In the paper [5] has been given two new criteria connected with AAC conjecture. In this purpose has been used the representation of $\sqrt{p}$ as a simple continued fraction.

It is easy to see that if the generators t, u are even, so $t = 2x_0, u = 2y_0$ the the equation (1.2) reduce to the equation

(1.5) $x_0^2 - py_0^2 = -1$.

In this paper we prove that AAC conjecure is true for even generators $t = 2x_0, u = 2y_0$.Namely we prove of the following theorem:

Theorem. If p is a prime number such that $p \equiv 1 \pmod 4$ and $\langle x_0, y_0\rangle$ is the fundamenal solution of the equqtio (1.5) then $p \nmid u$.

In the proof of the Theorem we use **Lemma 1** and **Lemma 2**, whoes been proved in our paper [5] as the Theorem1,Theorem 2 and Lemma 3.

2. Basic Lemmas

Lemma 1. Let p be a prime number such that $p \equiv 1 \pmod 4$ and let $p = b^2 + c^2, (b, c) = 1$. Moreover, let $\sqrt{p} = [q_0; \overline{q_1, q_2, ..., q_s}]$ be the representation of $\sqrt{p}$ as the simple continued fraction and let $\langle x_0, y_0\rangle$ be the fundamental solution of the equation (1.5) . Then $p \mid y_0$ if and only if

(2.1) $$p \mid cQ_r + bQ_{r-1} \text{ and } p \mid bQ_r - cQ_{r-1},$$

where $r = \frac{s-1}{2}$ and $\frac{P_n}{Q_n}$ is the $n-th$ convergent of the simple continued fraction of $\sqrt{p}$.

Lemma 2. Let be satisfied of the assumption of the Lemma 1. Then $p \mid y_0$ if and only if

$$p \mid 4bQ_rQ_{r-1} - (-1)^{r+1}\,. \tag{2.2}$$

Moreover, we have

$$P_{s-1} = P_rQ_r + P_{r-1}Q_{r-1}, \quad Q_{s-1} = Q_r^2 + Q_{r-1}^2. \tag{2.3}$$

3.Proof of the Theorem.

Suppose that $p \mid y_0$. For further consideration we use of the following well-known properties of the divisibility relation:

(R_1) if $d \mid a$ and $k \neq 0$, then $d \mid k \cdot a$,

(R_2) if $d \mid a$ and $d \mid b$, then $d \mid a + b$ and $d \mid a - b$,

where d, a, b, k are integer numbers.

From (R_1) and (2.1) it follows that $p \mid 4cQ_r^2 + 4bQ_rQ_{r-1}$. Hence, from (2.2) and the relation (R_2) we get

$$p \mid 4cQ_r^2 + (-1)^{r+1}\,. \tag{3.1}$$

In similar way from the second relation of (2.1), relation (2.2) and (R_2) we obtain,

$$p \mid 4cQ_{r-1}^2 - (-1)^{r+1}\,. \tag{3.2}$$

By (2.2) of Lemma 2 and (R_1) it follows that

$$p \mid 4bcQ_rQ_{r-1} - c\,(-1)^{r+1}\,. \tag{3.3}$$

On the other hand from (3.2) and (R_1) we have

$$p \mid 4bcQ_{r-1}^2 - b\,(-1)^{r+1}\,. \tag{3.4}$$

From (3.4),(3.3) and (R_2) we obtain

(3.5) $p \mid 4bcQ_{r-1}\left(Q_r+Q_{r-1}\right)-(-1)^{r+1}(b+c)$.

By completely similar way it follows that

(3.6) $p \mid 4bcQ_r^2+b\,(-1)^{r+1}$,

and

(3.7) $p \mid 4bcQ_r\left(Q_r-Q_{r-1}\right)+(-1)^{r+1}(b+c)$.

From (3.5) and the relation (R_1) with $k=Q_r-Q_{r-1}$ we obtain

(3.8) $p \mid 4bcQ_{r-1}\left(Q_r^2-Q_{r-1}^2\right)-(-1)^{r+1}(b+c)\left(Q_r-Q_{r-1}\right)$.

In similar way from the relation (R_1) with $k=Q_r+Q_{r-1}$ and (3.7) we get

(3.9) $p \mid 4bcQ_r\left(Q_r^2-Q_{r-1}^2\right)+(-1)^{r+1}(b+c)\left(Q_r+Q_{r-1}\right)$.

By (3.8),(3.9) and the relation (R_2) it follows that

(3.10) $p \mid 4bc\left(Q_r^2-Q_{r-1}^2\right)\left(Q_r-Q_{r-1}\right)+2\,(-1)^{r+1}Q_{r-1}(b+c)$.

We known that fundamental solution of the equation (1.5) is given by the formulas:

(F) $x_0=P_{s-1},\ \ y_0=Q_{s-1}$.

From (2.3) of Lemma 2 we have that $Q_{s-1}=Q_r^2+Q_{r-1}^2$. Hence, by the assumption that $p \mid y_0$ and second formula of (F) it follows that

(3.11) $p \mid Q_r^2+Q_{r-1}^2$.

It is easy to see that the following identity is true:

(3.12) $\left(Q_r^2-Q_{r-1}^2\right)\left(Q_r-Q_{r-1}\right)=\left(Q_r-Q_{r-1}\right)^2\left(Q_r+Q_{r-1}\right)=\left[\left(Q_r^2+Q_{r-1}^2\right)-2Q_rQ_{r-1}\right]\left(Q_r+Q_{r-1}\right)$.

From (3.12),(3.11),(3.10) and (3.1) we obtain

(3.13) $p \mid -8bcQ_rQ_{r-1}\left(Q_r+Q_{r-1}\right)+2\,(-1)^{r+1}Q_{r-1}(b+c)$.

By (2.2) of Lemma 2 we have that $p \nmid Q_{r-1}$ and consequently from well-known property of the divisibility relation and (3.13) we obtain

(3.14) $p \mid 4bcQ_r\left(Q_r + Q_{r-1}\right) - (-1)^{r+1}(b+c)$.

From (3.14) and (3.7) we get

(3.15) $p \mid 4bcQ_r\left(Q_r + Q_{r-1} + Q_r - Q_{r-1}\right)$.

The relation (3.15) implies that

(3.16) $p \mid 4bcQ_r^2$.

We observe that the relation (3.16) is impossible.In fact by (2.2) of Lemma 2 it follows that $p \nmid Q_r$ and consequently $p \nmid Q_r^2$. Since $p = b^2 + c^2$ and $(b,c) = 1$ then we have that $p \nmid b$ and $p \nmid c$.

Hence, we obtain a contradiction and the proof of the Theorem is complete.■

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Summary

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